MUTATIONS OF GROUP SPECIES WITH POTENTIALS AND THEIR REPRESENTATIONS. APPLICATIONS TO CLUSTER ALGEBRAS.

LAURENT DEMONET

ABSTRACT. This article tries to generalize former works of Derksen, Weyman and Zelevinsky about skew-symmetric cluster algebras to the skew-symmetrizable case. We introduce the notion of group species with potentials and their decorated representations. In good cases, we can define mutations of these objects in such a way that these mutations mimic the mutations of seeds defined by Fomin and Zelevinsky for a skew-symmetrizable exchange matrix defined from the group species. These good cases are called non-degenerate. Thus, when an exchange matrix can be associated to a non-degenerate group species with potential, we give an interpretation of the F-polynomials and the $\mathbf{g}$-vectors of Fomin and Zelevinsky in terms of the mutation of group species with potentials and their decorated representations. Hence, we can deduce a proof of a serie of combinatorial conjectures of Fomin and Zelevinsky in these cases. Moreover, we give, for certain skew-symmetrizable matrices a proof of the existance of a non-degenerate group species with potential realizing this matrix. On the other hand, we prove that certain skew-symmetrizable matrices can not be realized in this way.

CONTENTS

1. INTRODUCTION

The aim of this paper is to extend the results of [DWZ2] and [DWZ1] to the case of skew-symmetrizable exchange matrices. Unfortunately, the

techniques presented here do not work in any situation, but nevertheless in some important cases.

For this, we introduce *group species with potential* (GSP), which can be seen as quivers with potential with more than one idempotent at each vertex. Thus, we can also define a Jacobian ideal and a Jacobian algebra and study their representations. More precisely, we define the notion of a group species with potential with a *decorated representation* (GSPDR) and the notion of the mutation of a GSPDR at a vertex k (which is called the direction of the mutation). In *good cases*, we can mutate a GSPDR as many times as we want in any direction. In this case, the underlying GSP is called *non-degenerate*. Moreover, we can associate to certain GSPs, called locally free, a skew-symmetrizable matrix in such a way that the mutation we introduce projects, when it exists, to the mutation of matrix introduced by Fomin and Zelevinsky [FZ1]. Any skew-symmetrizable matrix can be reached in this way using a locally free GSP. The hard problem is to find which skew-symmetrizable matrix can be reached using a non-degenerate GSP. It is the case of matrices of the form DS where D is diagonal with positive integer coefficients and S is skew-symmetric with integer coefficients. It is also the case for the skew-symmetrizable matrices which occur in the situation of [Dem], in particular in all acyclic cases. Nevertheless, it is not always true, as shown by the counterexample at the end of section 12. The techniques presented in [DWZ2] work here almost in the same way. The only problem is that it is not always the case that for any 2-cycle, there exists a potential canceling it (this fact is very easy in the context of [DWZ2]).

We now explain the content of this article in more details. Let K be an algebraically closed field. Let I be a finite set and $E = \bigoplus_{i \in I} K[\Gamma_i]$ where, for each i, Γ_i is a finite group whose cardinal is not divisible by the characteristic of K. Let also A be an (E, E)-bimodule. This data is called a *group species* and its *complete path algebra* is

$$E\langle\langle A\rangle\rangle = \prod_{n \in \mathbb{N}} A^{\otimes n}.$$

A potential S on this group species can be seen as a (maybe infinite) linear combination of cyclic path, up to rotation. It permits to construct a two sided ideal $J(S)$, called the *Jacobian ideal* and a quotient algebra $\mathcal{P}(A, S) = E\langle\langle A\rangle\rangle/J(S)$ called the *Jacobian algebra*. A *decorated representation* of the GSP is a pair consisting of a $\mathcal{P}(A, S)$-module X and an E-module V. In sections 5 and 8, we define the mutation of a GSP with a decorated representation (GSPDR). This mutation is well defined if the group species has no loop and is 2-acyclic (that is, for any $i \in I$, $E_i(A \oplus A \otimes_E A)E_i = 0$, where $E_i = K[\Gamma_i] \subset E$).

In what follows, we suppose that the Γ_i are commutative and that the GSP is *locally free*, that is, for any $i, j \in I$, $E_i A E_j$ is a free E_i-left module and a free E_j-right module. In section 6, we define the exchange matrix B of a the group species by

$$b_{ij} = \dim_{E_j} A_{ji} - \dim_{E_j} A^*_{ij}.$$

Thus, the mutation of GSPDRs descends to the mutation of matrices defined by Fomin and Zelevinsky [FZ1]. In section 7, we discuss a class of matrices,

namely those of the form DS, for which there is always a non-degenerate GSP. Moreover, we remark that there exists also non-degenerate GSP in all cases which are categorified in [Dem] (because the endomorphisms rings of cluster-tilting objects constructed in [Dem] are Jacobian algebras). Remark also that there is no chance, with definitions given here, to construct non-degenerate GSPs for any skew-symmetrizable matrix, as shown by the counterexample ending section 12.

Following the ideas of [DWZ1], we explain in section 9 how to reinterpret the F-polynomials and $\mathbf{g}$-vectors defined in [FZ2] in terms of GSPDRs and their mutations. We deduce in section 11 that, when a skew-symmetrizable matrix can be obtained from a non-degenerate GSP, then the following conjectures are true:

Conjecture ([FZ2, conjecture 5.4])**.** *For any $\mathbf{i} \in I^n$ and $k \in I$, $F^B_{k;\mathbf{i}}$ has constant term* 1.

Conjecture ([FZ2, conjecture 5.5])**.** *For any $\mathbf{i} \in I^n$ and $k \in I$, $F^B_{k;\mathbf{i}}$ has a maximum monomial for divisibility order with coefficient* 1.

Conjecture ([FZ2, conjecture 7.12])**.** *For any $\mathbf{i} \in I^n$, $k \in I$, we denote by $k\mathbf{i}$ the concatenation of (k) and $\mathbf{i}$. Let $j \in I$ and $(g_i)_{i\in I} = \mathbf{g}^B_{j;\mathbf{i}}$ and $(g'_i)_{i\in I} = \mathbf{g}^{\mu_k(B)}_{j;k\mathbf{i}}$. Then we have, for any $i \in I$,*

$$g'_i = \begin{cases} -g_i & \text{if } i = k; \\ g_i + \max(0, b_{ik})g_k - b_{jk}\min(g_k, 0) & \text{if } i \neq k. \end{cases}$$

Conjecture ([FZ2, conjecture 6.13])**.** *For any $\mathbf{i} \in I^n$, the vectors $\mathbf{g}^B_{i;\mathbf{i}}$ for $i \in I$ are sign-coherent. In other terms, for $i, i', j \in I$, the j-th components of $\mathbf{g}^B_{i;\mathbf{i}}$ and $\mathbf{g}^B_{i';\mathbf{i}}$ have the same sign.*

Conjecture ([FZ2, conjecture 7.10(2)])**.** *For any $\mathbf{i} \in I^n$, the vectors $\mathbf{g}^B_{i;\mathbf{i}}$ for $i \in I$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^I$.*

Conjecture ([FZ2, conjecture 7.10(1)])**.** *For any $\mathbf{i}, \mathbf{i}' \in I^n$, if we have*

$$\sum_{i\in I} a_i \mathbf{g}^B_{i;\mathbf{i}} = \sum_{i\in I} a'_i \mathbf{g}^B_{i;\mathbf{i}'}$$

for some nonnegative integers $(a_i)_{i\in I}$ and $(a'_i)_{i\in I}$, then there is a permutation $\sigma \in \mathfrak{S}_I$ such that for every $i \in I$,

$$a_i = a'_{\sigma(i)} \quad \text{and} \quad a_i \neq 0 \Rightarrow \mathbf{g}^B_{i;\mathbf{i}} = \mathbf{g}^B_{\sigma(i);\mathbf{i}'} \quad \text{and} \quad a_i \neq 0 \Rightarrow F^B_{i;\mathbf{i}} = F^B_{\sigma(i);\mathbf{i}'}.$$

In particular, $F^B_{i;\mathbf{i}}$ is determined by $\mathbf{g}^B_{i;\mathbf{i}}$.

Thus, as stated in [FZ2, remark 7.11], if B is a full rank skew-symmetrizable matrix which correspond to a non-degenerate GSP, then the cluster monomials of a cluster algebra with exchange matrix B are linearly independent.

2. Group species and path algebras

Let K be a field.

Definition 2.1. A *group species* is a triple $(I, (\Gamma_i)_{i\in I}, (A_{ij})_{(i,j)\in I^2})$ where I is a finite set, for each $i \in I$, Γ_i is a finite group and for each $(i, j) \in I^2$, A_{ij} is a finite dimensional $(K[\Gamma_i], K[\Gamma_j])$-bimodule (the first acting on the left and the second on the right).

Fix now such a group species $Q = (I, (\Gamma_i)_{i\in I}, (A_{ij})_{(i,j)\in I^2})$

Definition 2.2. A *representation of* Q is a pair $((V_i)_{i\in I}, (x_{ij})_{(i,j)\in I^2})$ where for each $i \in I$, V_i is a right finite dimensional $K[\Gamma_i]$-module and for each $(i, j) \in I^2$,

$$x_{ij} \in \mathrm{Hom}_{\Gamma_j}(V_i \otimes_{\Gamma_i} A_{ij}, V_j).$$

Definition 2.3. Let $((V_i)_{i\in I}, (x_{ij})_{(i,j)\in I^2})$ and $((V_i')_{i\in I}, (x_{ij}')_{(i,j)\in I^2})$ be two representations of Q. A *morphism* from the first one to the second one is a family $(f_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_{\Gamma_i}(V_i, V_i')$ such that for each $(i, j) \in I^2$ the following diagram commute :

$$\begin{tikzcd} V_i \otimes_{\Gamma_i} A_{ij} \arrow[r, "x_{ij}"] \arrow[d, "f_i\otimes \mathrm{Id}_{A_{ij}}"'] & V_j \arrow[d, "f_j"] \\ V_i' \otimes_{\Gamma_i} A_{ij} \arrow[r, "x_{ij}'"'] & V_j' \end{tikzcd}$$

Remarks 2.4.
- The previous definitions give rise to an abelian category.
- If for each $i \in I$, Γ_i is the trivial group, we get back the classical definition of a quiver (up to the choice of a basis of each A_{ij}) and of the category of representations of a quiver.
- If for each $i \in I$, $K[\Gamma_i]$ is replaced by a division algebra, we obtain the usual definition of a species (see for example [DR]).

Definition 2.5. For each $i \in I$, denote $E_i = K[\Gamma_i]$. Denote also $E = \bigoplus_{i\in I} E_i$ and $A = \bigoplus_{(i,j)\in I^2} A_{ij}$. Thus, we put the natural (E, E)-bimodule structure on A and define the graded algebras

$$E\langle A\rangle = \bigoplus_{n\in\mathbb{N}} A^{\otimes n} \quad \text{and} \quad E\langle\langle A\rangle\rangle = \prod_{n\in\mathbb{N}} A^{\otimes n}$$

the first one being called the *path algebra* of the group species and the second one the *complete path algebra* of the group species (note that every tensor product is taken over E).

Remarks 2.6.
- As usual for quiver, the category of representations of a group species is equivalent to the category of finite dimensional right modules over its path algebra. Moreover, the category of nilpotent representations of a group species is equivalent to the category of finite dimensional right modules over its complete path algebra.
- If one denotes

  $$\mathfrak{m} = \prod_{n>0} A^{\otimes n} \subset E\langle\langle A\rangle\rangle$$

  which is clearly a two-sided ideal, then $E\langle\langle A\rangle\rangle$ becomes a topological algebra for the $\mathfrak{m}$-adic topology and $E\langle A\rangle$ is a dense subalgebra of it.

As in [DWZ2], $\mathfrak{m}$ is the unique maximal two-sided ideal of $E\langle\langle A\rangle\rangle$ not intersecting E. Moreover, if we have another group species with the same vertices whose arrows are encoded in the (E,E)-bimodule A', then, again as in [DWZ2], the morphisms φ from $E\langle\langle A\rangle\rangle$ to $E\langle\langle A'\rangle\rangle$ such that $\varphi|_E = \mathrm{Id}_E$ (later called E-morphisms) are parameterized in an obvious way by a pair $(\varphi^{(1)}, \varphi^{(2)})$ where $\varphi^{(1)} : A \to A'$ and $\varphi^{(2)} : A \to \mathfrak{m}'^2$ are (E,E)-bimodule morphisms. Thus, φ is an isomorphism if and only if $\varphi^{(1)}$ is an isomorphism. Introduce now the analogous of [DWZ2, definition 2.5]:

Definition 2.7. An E-automorphism φ of $E\langle\langle A\rangle\rangle$ will be called a *change of arrows* if $\varphi^{(2)} = 0$ and a *unitriangular automorphism* if $\varphi^{(1)} = \mathrm{Id}_A$.

Finally, introduce the following useful notation:

Notation 2.8. For all $i, j \in I$,

$$E\langle A\rangle_{ij} = E_i E\langle A\rangle E_j \quad \text{and} \quad E\langle\langle A\rangle\rangle_{ij} = E_i E\langle\langle A\rangle\rangle E_j$$

and for $n \in \mathbb{N}$,

$$A_{ij}^{\otimes n} = A^{\otimes n} \cap E\langle A\rangle_{ij} = A^{\otimes n} \cap E\langle\langle A\rangle\rangle_{ij}$$

so that

$$E\langle A\rangle_{ij} = \bigoplus_{n\in\mathbb{N}} A_{ij}^{\otimes n} \quad \text{and} \quad E\langle\langle A\rangle\rangle_{ij} = \prod_{n\in\mathbb{N}} A_{ij}^{\otimes n}.$$

3. Potential and their Jacobian ideals

Following [DWZ2] define:

Definition 3.1. Define

$$E\langle\langle A\rangle\rangle_{\mathrm{cyc}} = \frac{E\langle\langle A\rangle\rangle}{[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle]}$$

whose elements are called *potentials* (here, $[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle]$ is the closure of the two-sided ideal generated by commutators). As $[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle]$ is generated by its homogeneous elements, we can decompose $E\langle\langle A\rangle\rangle_{\mathrm{cyc}} = \prod_{n\in\mathbb{N}} A_{\mathrm{cyc}}^{\otimes n}$ where

$$A_{\mathrm{cyc}}^{\otimes n} = \frac{A^{\otimes n}}{[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle] \cap A^{\otimes n}}$$

and, if $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$, we write $S^{(n)}$ its summand which lies in $A_{\mathrm{cyc}}^{\otimes n}$.

Definition 3.2. Define the continuous linear map

$$\partial : (E\langle\langle A\rangle\rangle)^* \otimes_k E\langle\langle A\rangle\rangle \to E\langle\langle A\rangle\rangle$$

in the following way. First remark that $(E\langle\langle A\rangle\rangle)^* \simeq \bigoplus_{n\in\mathbb{N}} (A^{\otimes n})^*$. Then, if $\xi \in (A^{\otimes n})^*$ and $a_1, a_2, \dots, a_\ell \in A$ define $\partial_\xi(a_1 a_2 \dots a_\ell) = 0$ if $\ell < n$ and

$$\partial_\xi(a_1 a_2 \dots a_\ell) = \sum_{j=1}^{\ell} \sum_{g,h\in\mathcal{B}} \xi\left(g^{-1} a_j a_{j+1} \dots a_{j+n-1} h\right) h^{-1} a_{j+n} a_{j+n+1} \dots a_{j-1} g$$

if $\ell \geqslant n$ where all indices are taken modulo ℓ and $\mathcal{B} = \bigcup_{i\in I} \Gamma_i \subset E$. It is easy to see that ∂ is well defined and moreover that it vanishes on commutators. Thus, we can descend ∂ to a continuous linear map

$$\partial : (E\langle\langle A\rangle\rangle)^* \otimes_k E\langle\langle A\rangle\rangle_{\mathrm{cyc}} \to E\langle\langle A\rangle\rangle.$$

Remark 3.3. With the natural structure of (E, E)-bimodule on $(E\langle\langle A\rangle\rangle)^*$, one gets, for any $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$, that $\xi \mapsto \partial_\xi S$ is a morphism of (E, E)-bimodules.

Definition 3.4. For a potential $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$, define the *Jacobian ideal* $J(S)$ to be the closure of the two-sided ideal of $E\langle\langle A\rangle\rangle$ generated by the $\partial_\xi(S)$ for $\xi \in A^*$. The quotient $E\langle\langle A\rangle\rangle/J(S)$ is called the *Jacobian algebra* and is denoted by $\mathcal{P}(A, S)$ (we do not keep trace of $(I, (\Gamma_i))$ in the notation because it will be fixed).

Note that every E-morphism $\varphi : E\langle\langle A\rangle\rangle \to E\langle\langle A'\rangle\rangle$ descends to $\varphi : E\langle\langle A\rangle\rangle_{\mathrm{cyc}} \to E\langle\langle A'\rangle\rangle_{\mathrm{cyc}}$.

It is now easy to adapt the proof of [DWZ2, proposition 3.7]:

Proposition 3.5. *Let $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$. Every E-isomorphism $\varphi : E\langle\langle A\rangle\rangle \to E\langle\langle A'\rangle\rangle$ maps $J(S)$ to $J(\varphi(S))$ and therefore induces an isomorphism*

$$\mathcal{P}(A, S) \to \mathcal{P}(A', \varphi(S)).$$

4. Group species with potentials

For the rest of this article, the data $(I, (\Gamma_i))$ and so E will be fixed. Following the ideas of [DWZ2], define:

Definition 4.1. As before, A is an (E, E)-bimodule and we take $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$. We say that (A, S) is a group species with potential (GSP for short) if the species has no loop (for all $i \in I$, $E_iAE_i = \{0\}$) and $S \in \prod_{n>1} A^{\otimes n}_{\mathrm{cyc}}$.

Definition 4.2. Let (A, S) and (A', S') be two GSPs. One says that an E-isomorphism $\varphi : E\langle\langle A\rangle\rangle \to E\langle\langle A'\rangle\rangle$ is a right-equivalence if $\varphi(S) = S'$.

Note that this definition induces a equivalence relation. Moreover, a right equivalence $(A, S) \simeq (A', S')$ induces isomorphisms of (E, E)-bimodules $A \simeq A'$, $J(S) \simeq J(S')$ and $\mathcal{P}(A, S) \simeq \mathcal{P}(A', S')$ as said before.

Notation 4.3. If (A, S) and (A', S') are two GSPs, define $(A, S) \oplus (A', S') = (A \oplus A', S + S')$ so that $\mathcal{P}((A, S) \oplus (A', S'))$ is the completion of $\mathcal{P}(A, S) \oplus \mathcal{P}(A', S')$ for the product topology.

Definition 4.4. We say that a GSP (A, S) is *trivial* if $S \in A^{\otimes 2}_{\mathrm{cyc}}$ and $\{\partial_\xi(S) \mid \xi \in A^*\} = A$, or, equivalently, if $\mathcal{P}(A, S) = E$.

The following easy proposition is an adaptation of [DWZ2, proposition 4.4]:

Proposition 4.5. *A GSP (A, S) is trivial if and only if there exist an (E, E)-bimodule B and an (E, E)-bimodules isomorphism $\varphi : A \to B \oplus B^*$ such that*

$$\varphi(S) = \sum_{b \in \mathcal{B}} b \otimes b^*$$

where φ is naturally extended to an isomorphism $E\langle\langle A\rangle\rangle_{\mathrm{cyc}} \to E\langle\langle B \oplus B^*\rangle\rangle_{\mathrm{cyc}}$ and the right member does not depend of the choice of a basis $\mathcal{B}$ of B.

One gets also this proposition, similar to [DWZ2, proposition 4.5]:

Proposition 4.6. *If (A,S) is a GSP and (B,T) is a trivial GSP, then the canonical embedding $E\langle\langle A\rangle\rangle \hookrightarrow E\langle\langle A\oplus B\rangle\rangle$ induces an isomorphism $\mathcal{P}(A,S) \simeq \mathcal{P}(A\oplus B, S+T)$.*

For a GSP (A,S), we define the *trivial* and *reduced* part of A as the (E,E)-bimodules

$$A_{\text{triv}} = \{\partial_\xi S^{(2)} \mid \xi \in A^*\} \quad \text{and} \quad A_{\text{red}} = A/A_{\text{triv}}.$$

Moreover, we say that (A,S) is reduced if $S^{(2)} = 0$, or, equivalently, if $A_{\text{triv}} = \{0\}$.

Again, the proof of [DWZ2, theorem 4.6] is easy to adapt:

Theorem 4.7. *For any GSP (A,S), there exist $S_{\text{triv}} \in E\langle\langle A_{\text{triv}}\rangle\rangle$ and $S_{\text{red}} \in E\langle\langle A_{\text{red}}\rangle\rangle$ such that (A,S) is right equivalent to $(A_{\text{triv}}, S_{\text{triv}}) \oplus (A_{\text{red}}, S_{\text{red}})$.*

Moreover, the right equivalence classes of $(A_{\text{triv}}, S_{\text{triv}})$ and $(A_{\text{red}}, S_{\text{red}})$ are uniquely determined by the right equivalence class of (A,S).

Definition 4.8. A group species $(I, (\Gamma_i), A)$ is called 2-*acyclic* if, for any $i \in I$, $E_i A^{\otimes 2} E_i = \{0\}$.

We will see now how to find, as in [DWZ2], algebraic conditions guaranteeing the 2-acyclicity of the reduced part of a group species. Let $K\left[E\langle\langle A\rangle\rangle_{\text{cyc}}\right]$ be the ring of polynomial functions on $E\langle\langle A\rangle\rangle_{\text{cyc}}$ vanishing on all but a finite number of the $A^{\otimes n}_{\text{cyc}}$.

For each $S \in E\langle\langle A\rangle\rangle_{\text{cyc}}$ and $i,j \in I$, define the bilinear form $\alpha_{S,ij}$ by:

$$\begin{aligned} A^*_{ij} \times A^*_{ji} &\to K \\ (f,g) &\mapsto \sum_{\substack{\gamma\in\Gamma_i \\ \gamma'\in\Gamma_j}} \left[(\gamma' f \gamma^{-1} \otimes \gamma g \gamma'^{-1})\left(S^{(2)}\right) + (\gamma g \gamma'^{-1} \gamma' f \gamma^{-1})\left(S^{(2)}\right)\right]. \end{aligned}$$

First, an easy lemma:

Lemma 4.9. *Let $i,j \in I$. The followings are equivalent:*

(i) there exists $S \in E\langle\langle A\rangle\rangle_{\text{cyc}}$ such that $\alpha_{S,ij}$ is of maximal rank;
(ii) either A^*_{ij} is a subbimodule of A_{ji} or A^*_{ji} is a subbimodule of A_{ij}.

Proof. We clearly have $\alpha_{S,ij} = \alpha_{S,ji}$ for any S and therefore, one can suppose without loss of generality that $\dim_K A_{ij} \leqslant \dim_K A_{ji}$. Suppose that $\alpha_{S,ij}$ is of maximal rank. In any basis, the matrix of $\alpha_{S,ij}$ is the matrix of $A^*_{ij} \to A_{ji} : \xi \mapsto \partial_\xi(S^{(2)})$ and therefore, A^*_{ij} is a subbimodule of A_{ji}.

Reciprocally, suppose that A^*_{ij} is a subbimodule of A_{ji}. Thus, if $\mathcal{B}$ is a basis of A_{ij}, define

$$S = \sum_{a\in\mathcal{B}} a \otimes a^*$$

where $a^* \in A^*_{ij}$ is identified with its image in A_{ji}. Then, it is clear that $\alpha_{S,ij}$ is of maximal rank. □

Again, it is easy to generalize [DWZ2, proposition 4.15]:

Proposition 4.10. *The reduced part of a GSP (A,S) is 2-acyclic if and only if, for any $i,j \in I$, $\alpha_{S,ij}$ is of maximal rank. This condition is open. Moreover, if, for any $i,j \in I$, either A^*_{ij} is a subbimodule of A_{ji}, either*

A^*_{ji} is a subbimodule of A_{ij}, then there is a non empty Zariski open subset U of $E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$, a 2-acyclic (E,E)-bimodule A' and a regular map $H : U \to E\langle\langle A'\rangle\rangle_{\mathrm{cyc}}$ such that for any $S \in U$, $(A_{\mathrm{red}}, S_{\mathrm{red}})$ is right equivalent to $(A', H(S))$.

Proof. The arguments are the same than in [DWZ2]. For each $i, j \in I^2$, choose $\overline{A}^*_{ij} \subset A^*_{ij}$ such that $\overline{A}^*_{ij} = A^*_{ij}$ or $\overline{A}^*_{ij} \simeq A_{ji}$. Let U to be the non-empty open subset of $E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ containing the S such that for all $i, j \in I$, $\alpha_{S,ij}|_{\overline{A}^*_{ij}\times\overline{A}^*_{ji}}$ is non-degenerate (it corresponds to the non-vanishing of a fixed maximal minor of $\alpha_{S,ij}$). Define A' to be the intersection of the kernels of the elements of the $\overline{A}^*_{ij}$. Then the construction of H follows the proof of [DWZ2, theorem 4.6]. □

5. Mutations of group species with potential

Let (A, S) and $k \in I$ be a vertex such that $E_k A^{\otimes 2} E_k = \{0\}$ (we say that (A, S) is 2-acyclic at k). We suppose also that for any $i \in I$, the characteristic of K does not divide $\#\Gamma_i$. As in [DWZ2, §5], one defines $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ where, if $i, j \in I$,

$$\widetilde{A}_{ij} = \begin{cases} A^*_{ji} & \text{if } k \in \{i, j\}; \\ A_{ij} \oplus A_{ik} \otimes_{E_k} A_{kj} & \text{otherwise.} \end{cases}$$

In other terms,

$$\widetilde{A} = \overline{E}_k A \overline{E}_k \oplus A E_k A \oplus (E_k A)^* \oplus (A E_k)^*$$

where $\overline{E}_k = \bigoplus_{i \neq k} E_i$. Let now $[-] : \overline{E}_k E\langle\langle A\rangle\rangle \overline{E}_k \to E\langle\langle \widetilde{A}\rangle\rangle$ be the morphism of k-algebras generated by $[a] = a$ if $a \in \overline{E}_k A \overline{E}_k$ and $[ab] = ab \in A E_k A$ if $a \in A E_k$ and $b \in E_k A$ which is well defined because (A, S) has no loop. Again, because (A, S) has no loop, every potential $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ has a representative in $\overline{E}_k E\langle\langle A\rangle\rangle \overline{E}_k$ and it is easy to see that $[-]$ descends to a map

$$[-] : E\langle\langle A\rangle\rangle_{\mathrm{cyc}} \to E\langle\langle \widetilde{A}\rangle\rangle_{\mathrm{cyc}}.$$

Moreover, as for any $i \in I$ the characteristic of K does not divide $\#\Gamma_i$, we have a canonical sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_E (A E_k A, A E_k A) &\simeq (A E_k A)^* \otimes_E A E_k A \simeq (A E_k \otimes_E E_k A)^* \otimes_E A E_k A \\ &\simeq (E_k A)^* \otimes_E (A E_k)^* \otimes_E A E_k A \subset E\langle\langle \widetilde{A}\rangle\rangle \end{aligned}$$

and we define $\Delta_k(A)$ to be the image of $\mathrm{Id}_{A E_k A}$ through this isomorphism. Thus, define

$$\widetilde{S} = [S] + \Delta_k(A).$$

The proof of [DWZ2, proposition 5.1] can be easily generalized:

Proposition 5.1. *If (A', S') is another GSP such that $E_k A' = A' E_k = \{0\}$, then*

$$\widetilde{\mu}_k(A \oplus A', S + S') = \mu_k(A, S) \oplus (A', S').$$

Now, the proof of [DWZ2, theorem 5.2] is easy to generalize:

Theorem 5.2. *The right-equivalence class of the GSP $\widetilde{\mu}_k(A, S)$ is fully determined by the right-equivalence class of (A, S).*

Definition 5.3. Using theorem 5.2 together with theorem 4.7, the right-equivalence class of the reduced part of $\widetilde{\mu}_k(A,S)$ is fully determined by the right-equivalence class of (A,S). Thus we can define the map μ_k from the set of right-equivalence classes which are 2-acyclic at k to itself. It is called the *mutation at vertex k*.

Again, the proof of [DWZ2, theorem 5.7] is easy to generalize:

Theorem 5.4. *μ_k is an involution.*

Let us also remark that [DWZ2, proposition 6.1], [DWZ2, proposition 6.4] and [DWZ2, corollary 6.6] can be generalized:

Proposition 5.5. *The algebras $\overline{E}_k\mathcal{P}(A,S)\overline{E}_k$ and $\overline{E}_k\mathcal{P}\left(\widetilde{\mu}_k(A,S)\right)\overline{E}_k$ are isomorphic.*

Proposition 5.6. *The Jacobian algebra $\mathcal{P}(A,S)$ is finite-dimensional if and only if $\mathcal{P}\left(\widetilde{\mu}_k(A,S)\right)$ is.*

Corollary 5.7. *The Jacobian algebras $\overline{E}_k\mathcal{P}(A,S)\overline{E}_k$ and $\overline{E}_k\mathcal{P}\left(\mu_k(A,S)\right)\overline{E}_k$ are isomorphic and $\mathcal{P}(A,S)$ is finite-dimensional if and only if $\mathcal{P}\left(\mu_k(A,S)\right)$ is.*

As stated in [DWZ2, remark 6.8], the following definition makes sense:

Definition 5.8. We define the *deformation space of* (A,S) to be

$$\operatorname{Def}(A,S) = \frac{\mathcal{P}(A,S)}{E + [\mathcal{P}(A,S),\mathcal{P}(A,S)]}$$

where $[\mathcal{P}(A,S),\mathcal{P}(A,S)]$ is the closure of the two-sided ideal of $\mathcal{P}(A,S)$ generated by the commutators.

Thus, let us introduce the following extension of [DWZ2, proposition 6.9]:

Proposition 5.9. *We have an isomorphism:*

$$\operatorname{Def}(A,S) \simeq \operatorname{Def}\left(\widetilde{\mu}_k(A,S)\right).$$

Proof. It is enough to prove that

$$\frac{\overline{E}_k\mathcal{P}(A,S)\overline{E}_k}{\overline{E}_k + \left[\overline{E}_k\mathcal{P}(A,S)\overline{E}_k,\overline{E}_k\mathcal{P}(A,S)\overline{E}_k\right]} \hookrightarrow \operatorname{Def}(A,S)$$

is in fact an isomorphism (which is true because A has no loop) and to use proposition 5.5.

As in [DWZ2],

Definition 5.10. The GSP (A,S) is called *rigid* if $\operatorname{Def}(A,S) = \{0\}$.

Corollary 5.11. *The GSP (A,S) is rigid if and only if $\mu_k(A,S)$ is.*

6. Exchange matrices

We suppose now that A has neither loop nor 2-cycle (that is $A^{\otimes 1}_{\mathrm{cyc}} = A^{\otimes 2}_{\mathrm{cyc}} = \{0\}$). We suppose also that for any $(i,j) \in I^2$, A_{ij} is a free left E_i-module and a free right E_j-module (we will call it a *locally free* GSP). Define the matrix $B = B(A) = B(A,S)$ to be the matrix with rows and columns indexed by I and coefficients

$$b_{ij} = \dim_{E_j} A_{ji} - \dim_{E_j} A^*_{ij}$$

(by default, dimension are taken relatively to the left module structure). This matrix is clearly skew-symmetrizable since

$$\#\Gamma_j \times b_{ij} = \dim_K A_{ji} - \dim_K A^*_{ij}.$$

Definition 6.1. The matrix B is called the *exchange matrix* of A.

The following proposition justifies this generalization of [DWZ2]:

Proposition 6.2. *Every skew-symmetrizable matrix B can be reached in this way from a GSP.*

Proof. Let B be a skew-symmetrizable matrix and $D = (d_i)_{i \in I}$ be a diagonal matrix with positive integer coefficients such that BD is skew-symmetric. Let $\Gamma_i = \mathbb{Z}/d_i\mathbb{Z}$ and for $(i,j) \in I^2$ such that $b_{ij} > 0$,

$$A_{ji} = K\left[\mathbb{Z}/(d_j b_{ij})\mathbb{Z}\right] = K\left[\mathbb{Z}/(-d_i b_{ji})\mathbb{Z}\right]$$

which is a left and right free (Γ_j, Γ_i)-bimodule. It is clear that $A = \bigoplus_{i,j \in I} A_{ij}$ has exchange matrix B. □

Proposition 6.3. *Let $k \in I$.*

(i) The GSP $\widetilde{\mu}_k(A,S)$ is locally free.
(ii) If $\mu_k(A,S)$ is 2-acyclic then it is locally free.
(iii) If $\mu_k(A,S)$ is 2-acyclic then

$$\mu_k(B(A,S)) = B(\mu_k(A,S))$$

where the μ_k on the left hand is the one defined in [FZ1]. *Namely:*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} + \dfrac{b_{ik}\,|b_{kj}| + |b_{ik}|\,b_{kj}}{2} & \text{otherwise} \end{cases}$$

if $B' = \mu_k(B)$.

Proof. (i) First of all, it is clear that for $i \in I$, $E_i^* \simeq E_i$ as (E_i, E_i)-bimodules (as E_i is finite dimensional). Thus, for any i, A^*_{ik} and A^*_{ki} are left and right free modules. Moreover, as a right module,

$$A_{ik} \otimes_{E_k} A_{kj} \simeq A_{kj}^{\dim_{E_k}(A^*_{ik})}$$

and, as a left module,

$$A_{ik} \otimes_{E_k} A_{kj} \simeq A_{ik}^{\dim_{E_k}(A_{kj})}$$

which ends the proof that $\widetilde{\mu}_k(A,S)$ is locally free.

(ii) If one denotes $(\widetilde{A}, \widetilde{S}) = \widetilde{\mu}_k(A, S)$, one has
$$\widetilde{A} = \widetilde{A}_{\mathrm{red}} \oplus \widetilde{A}_{\mathrm{triv}}$$
As $\widetilde{A}_{\mathrm{red}}$ is 2-acyclic, for any $i, j \in I$, $\widetilde{A}_{\mathrm{red},ij} = 0$ or $\widetilde{A}_{\mathrm{red},ji} = 0$. Suppose that $\widetilde{A}_{\mathrm{red},ij} = 0$. Hence $\widetilde{A}_{\mathrm{triv},ji} \simeq \widetilde{A}^*_{\mathrm{triv},ij} \simeq \widetilde{A}^*_{ij}$ is left and right free (thanks to the previous point). Moreover, $\widetilde{A}_{ji} = \widetilde{A}_{\mathrm{red},ji} \oplus \widetilde{A}_{\mathrm{triv},ji}$ and, as the categories of left E_j-modules and right E_i-modules are Krull-Schmidt, $\widetilde{A}_{\mathrm{red},ji}$ is left and right free.

(iii) It is enough to remark that
$$\dim_{E_i} A_{ik} \otimes_{E_k} A_{kj} = \dim_{E_i} A_{ik}^{\dim_{E_k} A_{kj}} = \dim_{E_i}(A_{ik}) \dim_{E_k}(A_{kj})$$
and that
$$\dim_{E_i}(A_{jk} \otimes_{E_k} A_{ki})^* = \dim_{E_i} (A^*_{ki})^{\dim_{E_k} A^*_{jk}} = \dim_{E_i}(A^*_{ki}) \dim_{E_k}(A^*_{jk})$$
and to use the definition and the duality $A_{\mathrm{triv},ij} \simeq A^*_{\mathrm{triv},ji}$. □

Definition 6.4. The group species is said to be *globally free* if, for any $i, j \in I$, A_{ij} is a free (E_i, E_j)-bimodule (that is a free $E_i \otimes_K E_j^{\mathrm{op}}$-module).

Remark 6.5. The class of globally free group species is stable under mutation.

Proposition 6.6. *If a matrix is of the form DB, where D is diagonal with positive integer coefficients and B is skew-symmetric, then the group species constructed in proposition 6.2 is globally free.*

7. Existance of nondegenerate potentials

If $(I, (\Gamma_i), A)$ is a group species without loop nor 2-cycle, a potential $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ will be said to be *non-degenerate* if every sequence of mutation going from (A, S) yields to a 2-acyclic GSP.

We cite the following adapted result, whose proof is the same than the proof of [DWZ2, corollary 7.4]:

Theorem 7.1. *If the group species is globally free then there is a countable number of non-constant polynomials in $K\left[E\langle\langle A\rangle\rangle_{\mathrm{cyc}}\right]$ such that the non-vanishing of these polynomials on $S \in E\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ implies that S is non-degenerate. In particular if K is uncountable, there exist non-degenerate potentials.*

Proof. The only thing to change is that, if the group species is globally free, then for each $i, j \in I$, either A^*_{ij} is a subbimodule of A_{ji}, or A^*_{ji} is a subbimodule of A_{ij} and, therefore, proposition 4.10 can be applied. □

Remark 7.2. It is also easy to prove that for any skew-symmetrizable matrix B coming from the categories with an action of a group Γ considered in [Dem], there is a non-degenerate GSP realizing it. More precisely, the endomorphism ring of a Γ-stable cluster-tilting object in the stable category of a category constructed in [Dem] can be realized by a non-degenerate GSP (it is the case because Γ-2-cycles do not appear after mutations). In particular, the only potential for an acyclic group species is non-degenerate.

Another proposition linking rigid and non-degenerate potentials can be adapted from [DWZ2, proposition 8.1 and corollary 8.2]:

Proposition 7.3. *Every rigid globally free GSP (A, S) is 2-acyclic and, in this case, S is non-degenerate.*

As in [DWZ2, §8], there exist group species without rigid potentials. The techniques of [DWZ2, §8] work also in the context of this article.

8. Decorated representations and their mutations

The aim of this section is to adapt the results of [DWZ2, §10]. We suppose here that for any $i \in I$, the characteristic of K does not divide the cardinal of Γ_i.

Following [DWZ2, definition 10.1],

Definition 8.1. A *decorated representation* of a GSP (A, S) is a pair (X, V) where X is a $\mathcal{P}(A, S)$-module and V is a E-module.

In the following, we will look at pairs consisting of a GSP (A, S) and a decorated representation of it. We will denote this type of objects by (A, S, X, V) and call them *group species with potential and decorated representation* (GSPDR).

Following [DWZ2, definition 10.2],

Definition 8.2. A right-equivalence between two GSPDRs (A, S, X, V) and (A', S', X', V') is a triple (φ, ψ, η) such that:

- $\varphi : E\langle\langle A\rangle\rangle \to E\langle\langle A'\rangle\rangle$ is a right-equivalence from (A, S) to (A', S') (see definition 4.2);
- $\psi : X \to X'$ is a linear isomorphism such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "u_X"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ X' \arrow[r, "\varphi(u)_{X'}"'] & X' \end{tikzcd}$$

  for any $u \in E\langle\langle A\rangle\rangle$;
- $\eta : V \to V'$ is an isomorphism.

Using proposition 4.6, for each GSPDR (A, S, X, V), the decorated representation (X, V) can be seen as a representation of $(A_{\text{red}}, S_{\text{red}})$. Thus, we can call $(A_{\text{red}}, S_{\text{red}}, X, V)$ the *reduced part* of (A, S, X, V). As in [DWZ2, proposition 10.5], the right-equivalence class of the reduced part of a GSPDR is fully determined by the right-equivalence class of this GSPDR.

Now, we can define the mutation of a GSPDR (A, S, X, V). Let $k \in I$. Our aim is to define a GSPRD $\mu_k(A, S, X, V) = (A', S', X', V')$ such that $(A', S') = \mu_k(A, S)$. Denote:

$$X_{\text{in}} = X \otimes_E AE_k \quad \text{and} \quad X_{\text{out}} = X \otimes_E A^* E_k.$$

Thus, we can define two right E_k-module morphisms. One, α, from X_{in} to $X_k = XE_k$ which is the application $(x \otimes a) \mapsto xa$ and one from X_k to X_{out} which is defined by

$$\beta(x) = \sum_{b \in \mathcal{B}} xb \otimes b^*$$

which does not depend on the basis $\mathcal{B}$ of E_kA. Observe also that we have a canonical sequence of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{E_k}(X_{\mathrm{out}}, X_{\mathrm{in}}) &\simeq \mathrm{Hom}_E(X \otimes_E A^*E_k \otimes_{E_k} E_kA^*, X)\\ &\simeq \mathrm{Hom}_E(X \otimes_E (AE_kA)^*, X)\end{aligned}$$

It is not hard to see that $[x \otimes \xi \mapsto x(\partial_\xi S)] \in \mathrm{Hom}_E(X \otimes_E (AE_kA)^*, X)$. Let γ be the corresponding element of $\mathrm{Hom}_{E_k}(X_{\mathrm{out}}, X_{\mathrm{in}})$.

So we get, as in [DWZ2] a commutative diagram of right E_k-modules:

$$\begin{tikzcd} & X_k \arrow[dr, "\beta"] & \\ X_{\mathrm{in}} \arrow[ur, "\alpha"] & & X_{\mathrm{out}} \arrow[ll, "\gamma"] \end{tikzcd}$$

with $\alpha\gamma = \gamma\beta = 0$ [DWZ2, lemma 10.6]. For $i \in I$, define:

$$X'_i = \begin{cases} X_i & \text{if } i \neq k \\ \dfrac{\ker \gamma}{\mathrm{im}\, \beta} \oplus \mathrm{im}\, \gamma \oplus \dfrac{\ker \alpha}{\mathrm{im}\, \gamma} \oplus V_i & \text{if } i = k \end{cases}$$

and

$$V'_i = \begin{cases} V_i & \text{if } i \neq k \\ \dfrac{\ker \beta}{\ker \beta \cap \mathrm{im}\, \alpha} & \text{if } i = k \end{cases}$$

To get the structure of an $\mathcal{P}(A', S')$-module on X', we must define the way $\widetilde{A}$ acts on it where $(\widetilde{A}, \widetilde{S}) = \widetilde{\mu}_k(A, S)$ (as $\mathcal{P}(A', S') \simeq \mathcal{P}(\widetilde{A}, \widetilde{S})$). Recall from §5, that

$$\widetilde{A} = \overline{E}_k A \overline{E}_k \oplus AE_kA \oplus (E_kA)^* \oplus (AE_k)^*.$$

First of all, $\overline{E}_k A \overline{E}_k \oplus AE_kA \subset \overline{E}_k E\langle\langle A\rangle\rangle \overline{E}_k$ and for the vertices outside k, $X'_k = X_k$. Therefore, we can take the same action for this part of $\widetilde{A}$. For the rest, we have $\widetilde{A}E_k = A^*E_k$ and $\widetilde{A}^*E_k = AE_k$ and therefore, we have to define:

$$\alpha' : X'_{\mathrm{in}} = X' \otimes_E \widetilde{A}E_k = X \otimes_E A^*E_k = X_{\mathrm{out}} \to X'_k$$

and

$$\beta' : X'_k \to X'_{\mathrm{out}} = X' \otimes_E \widetilde{A}^*E_k = X \otimes_E AE_k = X_{\mathrm{in}}$$

As in [DWZ2], we have to choose a *splitting data*:

- let $\rho : X_{\mathrm{out}} \twoheadrightarrow \ker \gamma$ be a splitting of $\ker \gamma \hookrightarrow X_{\mathrm{out}}$ in the category $\mathrm{mod}\, E_k$ (it is possible, as the characteristic of K does not divide the cardinal of Γ_k);
- let $\sigma : \ker \alpha / \mathrm{im}\, \gamma \hookrightarrow \ker \alpha$ a splitting of $\ker \alpha \twoheadrightarrow \ker \alpha / \mathrm{im}\, \gamma$ in $\mathrm{mod}\, E_k$.

Now, using the direct sum decomposition

$$X'_k = \frac{\ker \gamma}{\mathrm{im}\, \beta} \oplus \mathrm{im}\, \gamma \oplus \frac{\ker \alpha}{\mathrm{im}\, \gamma} \oplus V_i,$$

define

$$\alpha' = \begin{pmatrix} -\pi\rho \\ -\gamma \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta' = \begin{pmatrix} 0 & \iota & \iota\sigma & 0 \end{pmatrix}$$

where π designs the canonical projection and ι the canonical injections.

Again, [DWZ2, proposition 10.7] can be adapted:

Proposition 8.3. *The above definition gives rise to a decorated representation of $(\widetilde{A}, \widetilde{S})$ and, therefore, through the isomorphism $\mathcal{P}(\widetilde{A}, \widetilde{S}) \simeq \mathcal{P}(A', S')$, to a decorated representation of (A', S').*

Notation 8.4. We denote

$$\widetilde{\mu}_k(A, S, X, V) = (\widetilde{A}, \widetilde{S}, X', V') \quad \text{and} \quad \mu_k(A, S, X, V) = (A', S', X', V').$$

We can adapt [DWZ2, proposition 10.9]:

Proposition 8.5. *The isomorphism class of the GSPDR $\widetilde{\mu}_k(A, S, X, V)$ does not depend on the choice of the splitting data.*

and [DWZ2, proposition 10.10 and corollary 10.12]:

Proposition 8.6. *The right-equivalence classes of the GSPDRs*

$$\widetilde{\mu}_k(A, S, X, V) \quad \text{and} \quad \mu_k(A, S, X, V)$$

depend only on the right-equivalence class of (A, S, X, V).

Now an important theorem whose proof is the same as the one of [DWZ2, theorem 10.13]:

Theorem 8.7. *On the right-equivalence classes of GSPDRs which are 2-acyclic at k, μ_k is an involution.*

It is easy to define the notion of a direct sum of two decorated representations of a GSP and, therefore, the notion of an indecomposable decorated representation of a GSP. Thus, as μ_k clearly commutes with this type of direct sums, μ_k acts on GSPs with indecomposable decorated representations. We call a GSPDR (A, S, X, V) positive if $V = \{0\}$ and negative if $X = \{0\}$. Moreover, it is called *simple* at $i \in I$ if $X \oplus V$ is an indecomposable E_i-module. Then we adapt [DWZ2, proposition 10.15]:

Proposition 8.8. *An indecomposable GSPDR is either positive, or negative simple. The mutation μ_k exchange a positive simple at k with the corresponding negative simple at k. Moreover, it is the only case where a mutation interchanges positive and negative indecomposable GSPDRs.*

As in [DWZ1, §6], denote, for $k \in I$ and $X, X' \in \operatorname{mod} \mathcal{P}(A, S)$,

$$\operatorname{Hom}^{[k]}_{\mathcal{P}(A,S)}(X, X') = \left\{ f \in \operatorname{Hom}_{\mathcal{P}(A,S)}(X, X') \mid f|_{X\overline{E}_k} = 0 \right\}.$$

Cite now easy to adapt [DWZ1, proposition 6.1]:

Proposition 8.9. *The mutation μ_k induces an isomorphism*

$$\frac{\operatorname{Hom}_{\mathcal{P}(A,S)}(X, X')}{\operatorname{Hom}^{[k]}_{\mathcal{P}(A,S)}(X, X')} \simeq \frac{\operatorname{Hom}_{\mathcal{P}(\mu_k(A,S))}(\mu_k(X), \mu_k(X'))}{\operatorname{Hom}^{[k]}_{\mathcal{P}(\mu_k(A,S))}(\mu_k(X), \mu_k(X'))}.$$

Remark 8.10. As claimed in [DWZ1, §6], the isomorphism of proposition 8.9 can be seen as a functorial isomorphism by introducing adapted quotient categories.

9. F-POLYNOMIALS AND $\mathbf{g}$-VECTORS OF DECORATED REPRESENTATIONS

The aim of this section is to define the notions of the F-polynomial and the $\mathbf{g}$-vector of a GSPDR and to give a link with the usual notion (see [FZ2]). It is an extension of [DWZ1]. As before, $(I, (\Gamma_i))$ and therefore E are fixed. We suppose also that the characteristic of K does not divide any of the cardinals of the groups Γ_i. We suppose moreover that K is algebraically closed and that all the Γ_i are commutative (as seen in section 6, this case is sufficient to realize skew-symmetrizable exchange matrices).

Notation 9.1. For any $i \in I$, denote $\mathrm{irr}_i = \mathrm{irr}(\Gamma_i)$ the set of isomorphism classes of irreducible representations of Γ_i. One defines $\mathrm{irr} = \bigcup_{i \in I} \{i\} \times \mathrm{irr}_i$ and for $i \in I$, $C_i = K_0(\Gamma_i) \simeq \mathbb{Z}^{\mathrm{irr}_i}$. We also denote $C = K_0(E) = \bigoplus_{i \in I} C_i \simeq \mathbb{Z}^{\mathrm{irr}}$. If $V \in \operatorname{mod} E$ (resp. $V \in \operatorname{mod} E_i$), $[V]$ is its class in C (resp. in C_i). If $\mathbf{e} \in C$ (resp. $\mathbf{e} \in C_i$) and $(j, \rho) \in \mathrm{irr}$ (resp. $\rho \in \mathrm{irr}_i$) then $\mathbf{e}_{j,\rho}$ (resp. $\mathbf{e}_\rho$) is the coefficient of (j, ρ) (resp. ρ) in $\mathbf{e}$.

If $(Y_j)_{j \in \mathrm{irr}}$ (resp. $(Y_j)_{j \in \mathrm{irr}_i}$) is a family of indeterminates or of elements of a ring, and $\mathbf{e} \in C$ (resp. $\mathbf{e} \in C_i$), one denotes

$$Y^{\mathbf{e}} = \prod_{\substack{j \in \mathrm{irr} \\ (\text{resp. } j \in \mathrm{irr}_i)}} Y_j^{\mathbf{e}_j}.$$

If (A, S) is a GSP, X a representation of it, $[X]$ is its class, seen as an E-module, in C. If $\mathbf{e} \in C$ then $\mathrm{Gr}_{\mathbf{e}}(X)$ is the Grassmanian of the $\mathcal{P}(A, S)$-submodules X' of X such that $[X'] = \mathbf{e}$.

Let (A, S, X, V) be a GSPDR, we recall the diagram of section 8, by changing a little the notation:

$$\begin{tikzcd} & X(k) \arrow[dr, "\beta_k"] & \\ X_{\mathrm{in}}(k) \arrow[ur, "\alpha_k"] & & X_{\mathrm{out}}(k) \arrow[ll, "\gamma_k"] \end{tikzcd}$$

Definition 9.2. One defines the F-polynomial F_X of X to be a polynomial in $\mathbb{Z}\left[(Y_i)_{i \in \mathrm{irr}}\right]$ defined by:

$$F_X(Y) = \sum_{\mathbf{e} \in C} \chi\left(\mathrm{Gr}_{\mathbf{e}}(X)\right) Y^{\mathbf{e}}$$

where χ is the Euler characteristic. One define also the $\mathbf{g}$-vector $\mathbf{g}_{X,V} = (g_k)_{k \in I} \in C = \bigoplus_{k \in I} C_k$ by

$$g_k = [\ker \gamma_k] - [X(k)] + [V(k)].$$

With the same indexing, define $\mathbf{h}_{X,V} = (h_k)_{k \in I}$ by

$$h_k = -[\ker \beta_k].$$

Notation 9.3. If (Y) is a family of indeterminates, we denote by $\mathbb{Q}_+(Y)$ the free commutative semifield generated by its elements. If (y) is a family of elements of a commutative semifield, we denote by $\mathbb{Q}_+(y)$ the subsemifield generated by its elements.

Then, it is easy to adapt [DWZ1, proposition 3.1], [DWZ1, proposition 3.2] and [DWZ1, proposition 3.3]:

Proposition 9.4. *The polynomial $F_X(Y)$ has constant term 1 and maximum term (for divisibility of monomials) $Y^{[X]}$.*

Proposition 9.5. *If X' is another $\mathcal{P}(A,S)$-module then $F_{X\oplus X'} = F_X F_{X'}$.*

Proposition 9.6. *If $F_X \in \mathbb{Q}_+(Y)$, then F_X can by evaluated in the semifield $\operatorname{Trop}(Y')$ where $(Y')_{i\in\mathrm{irr}}$ is a family of indeterminates. Then $\mathbf{h}_X$ and F_X are related by the following formula:*

$$Y'^{\mathbf{h}_X} = F_X|_{\operatorname{Trop}(Y')}\left(Y'^{-1}_{i,\rho}Y'^{[\rho\otimes_{E_i}E_iA^*]}\right)_{(i,\rho)\in\mathrm{irr}}.$$

Proof. We follow the proof of [DWZ1]. Remark that for any $\mathbf{e}\in C$,

$$(Y^{\mathbf{e}})|_{\operatorname{Trop}(Y')}\left(Y'^{-1}_{i,\rho}Y'^{[\rho\otimes_{E_i}E_iA^*]}\right)_{(i,\rho)\in\mathrm{irr}} = Y'^{-\mathbf{e}+[\mathbf{e}\otimes_E A^*]}.$$

For $i\in I$, the exponent of $Y'_i = (Y_{i,\rho})_{\rho\in\mathrm{irr}_i}$ can be rewritten as

$$-\mathbf{e}_i + [\mathbf{e}\otimes_E A^*E_i]$$

which can be interpreted as

$$-[X'(i)] + [X'_{\mathrm{out}}(i)]$$

for any submodule X' of X such that $[X'] = \mathbf{e}$. Thus, the end of the proof is the same as in [DWZ1]. $\square$

Recall the definition of a Y-seed:

Definition 9.7 ([DWZ1, §2])**.** A *Y-seed* is a pair (y,B) where y is a family of elements of a semifield indexed by I and B is a skew-symmetrizable matrix. For $k\in I$, we define $\mu_k(y,B) = (y',\mu_k(B))$ where, for $i\in I$,

$$y'_i = \begin{cases} y_i^{-1} & \text{if } i = k \\ y_i y_k^{\max(0,b_{ki})}(1+y_k)^{-b_{ki}} & \text{if } i \neq k. \end{cases}$$

Now, define the notion of an extended Y-seed:

Definition 9.8. A *extended Y-seed* is a pair $(y,(A,S))$ where y is a family of elements of a semifield indexed by irr and (A,S) is a non-degenerate GSP. For $k\in I$, we define $\mu_k(y,(A,S)) = (y',\mu_k(A,S))$ where, for $(i,\rho)\in\mathrm{irr}$,

$$y'_{i,\rho} = \begin{cases} y_{i,\rho}^{-1} & \text{if } i = k \\ y_{i,\rho}y_k^{[\rho\otimes_{E_i}A_{ik}]}(1+y_k)^{[\rho\otimes_{E_i}A^*_{ki}]-[\rho\otimes_{E_i}A_{ik}]} & \text{if } i \neq k. \end{cases}$$

Remark 9.9. The mutation of extended Y-seeds is involutive.

Definition 9.10. A Y-seed or an extended Y-seed will be called *free* if its variables y are algebraically independent.

Remark 9.11. The notion of freeness for a Y-seed (or an extended Y-seed) is stable under mutations. The semifield $\mathbb{Z}_+(y)$ and the algebra $\mathbb{Z}[y]$ are also stable under mutation, as the mutation is involutive.

Definition 9.12. Let $(y, (A, S))$ be a free extended Y-seed and $(z, B(A))$ be a Y-seed (for the same A). The following morphism of algebra is called the *specialization map*:

$$\begin{aligned} \Phi_{y \to z} : \mathbb{Z}_+(y) &\to \mathbb{Z}_+(z) \\ y_{i,\rho} &\mapsto z_i. \end{aligned}$$

The analogous for $\mathbb{Z}[y]$ and $\mathbb{Z}[z]$ is also denoted by Φ.

Proposition 9.13. *Let $(y, (A, S))$ be a free extended Y-seed such that (A, S) is a locally free GSP, and $(z, B(A))$ be a Y-seed. Let $k \in I$. Denote $y' = \mu_k(y)$, and $z' = \mu_k(z)$. Then, $\Phi_{y' \to z'} = \Phi_{y \to z}$.*

Proof. As y' generates $\mathbb{Z}_+(y') = \mathbb{Z}_+(y)$, it is enough to look at this for the $y'_{i,\rho}$ for $(i, \rho) \in \mathrm{irr}$. By definition,

$$\Phi_{y'->z'}\left(y'_{i,\rho}\right) = z'_i$$

If $i = k$, then

$$\Phi_{y->z}\left(y'_{i,\rho}\right) = \Phi_{y->z}\left(y_{i,\rho}^{-1}\right) = z_i^{-1} = z'_i.$$

If $i \neq k$, then

$$\begin{aligned} \Phi_{y->z}\left(y'_{i,\rho}\right) &= \Phi_{y->z}\left(y_{i,\rho} y_k^{[\rho \otimes_{E_i} A_{ik}]} (1 + y_k)^{[\rho \otimes_{E_i} A^*_{ki}] - [\rho \otimes_{E_i} A_{ik}]}\right) \\ &= z_i \prod_{\sigma \in C_k} \left[z_k^{[\rho \otimes_{E_i} A_{ik}]_\sigma} (1 + z_k)^{[\rho \otimes_{E_i} A^*_{ki}]_\sigma - [\rho \otimes_{E_i} A_{ik}]_\sigma} \right] \\ &= z_i \left[z_k^{\dim_K(\rho \otimes_{E_i} A_{ik})} (1 + z_k)^{\dim_K(\rho \otimes_{E_i} A^*_{ki}) - \dim_K(\rho \otimes_{E_i} A_{ik})} \right] \\ &= z_i \left[z_k^{\dim_{E_i} A_{ik}} (1 + z_k)^{\dim_{E_i} A^*_{ki} - \dim_{E_i} A_{ik}} \right] \\ &= z_i \left[z_k^{\max(0, b_{ki})} (1 + z_k)^{-b_{ki}} \right] = z'_i \end{aligned}$$

(here we use the fact that every considered irreducible representation is of dimension 1, as the considered groups are commutative and K is algebraically closed). □

To make the relation with F-polynomials and $\mathbf{g}$-vectors in cluster algebras, we need the following adaptation of [DWZ1, lemma 5.2]:

Proposition 9.14. *Let (A, S, X, V) be a GSPDR such that (A, S) is non-degenerate. Let $k \in I$. Denote $(A', S', X', V') = \mu_k(A, S, X, V)$. Suppose also that the extended Y-seed $(y', (A', S'))$ is obtained from $(y, (A, S))$ by the mutation at k. Denote $\mathbf{g}_{X,V} = (g_i)_{i \in I}$, $\mathbf{g}_{X',V'} = (g'_i)_{i \in I}$, $\mathbf{h}_{X,V} = (h_i)_{i \in I}$ and $\mathbf{h}_{X',V'} = (h'_i)_{i \in I}$. Then*

(i) $\mathbf{g}_{X,V} = \mathbf{h}_{X,V} - \mathbf{h}_{X',V'}$;

(ii) one has

$$(y_k + 1)^{h_k} F_X(y) = (y'_k + 1)^{h'_k} F_{X'}(y')$$

where

$$(y_k+1)^{h_k} = \prod_{i\in \mathrm{irr}_k} (y_{(k,i)}+1)^{h_{ki}};$$

(iii) for any $j \in I$,

$$g'_j = \begin{cases} -g_j & \text{if } j = k \\ g_j + [g_k \otimes_{E_k} A_{kj}] - [h_k \otimes_{E_k} A_{kj}] + \left[h_k \otimes_{E_k} A^*_{jk}\right] & \text{if } j \neq k. \end{cases}$$

Proof. (i) By definition, for $i \in I$, $g_i = [\ker \gamma_i] - [X(i)] + [V(i)]$, $h_i = -[\ker \beta_i]$ and $h'_i = -[\ker \beta'_i]$ (where β' is the analogous of β for (X', V')). So it is enough to prove that

$$[\ker \gamma_i] + [V_i] + [\ker \beta_i] = [X(i)] + [\ker \beta'_i].$$

From the definition of β'_i given in section 8, it is easy to see that $\ker \beta'_i \simeq \ker(\gamma_i)/\operatorname{im}(\beta_i) \oplus V_i$. And, therefore, the searched equality reduces to

$$[\operatorname{im} \beta_i] + [\ker \beta_i] = [X(i)]$$

which is obvious.

(ii) We follow the proof of [DWZ1, lemma 5.2]. Let $\mathbf{e} \in C$ and $\mathbf{e}'$ its projection in $\bigoplus_{i\neq k} C_i$. Let $X_0 = X\overline{E}_k$ which is a $\overline{E}_k\mathcal{P}(A,S)\overline{E}_k$-module. For any $\overline{E}_k\mathcal{P}(A,S)\overline{E}_k$-submodule W of X_0, one can define

$$W_{\mathrm{in}}(k) = W \otimes_{\overline{E}_k} AE_k \subset X_{\mathrm{in}}(k) \quad \text{and} \quad W_{\mathrm{out}}(k) = W \otimes_{\overline{E}_k} A^*E_k \subset X_{\mathrm{out}}(k)$$

which are well defined because (A,S) has no loop (and therefore $X_{\mathrm{in}} = X \otimes_{\overline{E}_k} AE_k$ and $X_{\mathrm{out}} = X \otimes_{\overline{E}_k} A^*E_k$).

For $\mathbf{r}, \mathbf{s} \in C_k$, define $Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)$ to be the subvariety of $\mathrm{Gr}_{\mathbf{e}'}(X_0)$ consisting of the W satisfying

- $[\alpha_k(W_{\mathrm{in}}(k))] = \mathbf{r}$;
- $\left[\beta_k^{-1}(W_{\mathrm{out}}(k))\right] = \mathbf{s}$;
- $\alpha_k(W_{\mathrm{in}}(k)) \subset \beta_k^{-1}(W_{\mathrm{out}}(k))$.

Define also the variety

$$\widetilde{Z}_{\mathbf{e},\mathbf{r},\mathbf{s}}(X) = \left\{W \in \mathrm{Gr}_{\mathbf{e}}(X) \mid W\overline{E}_k \in Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right\}$$

so that, by an easy computation, $\widetilde{Z}_{\mathbf{e},\mathbf{r},\mathbf{s}}(X)$ is a fiber bundle over $Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)$ with fiber $\mathrm{Gr}_{e_k-\mathbf{r}}(\mathbf{s}-\mathbf{r})$ (where, by abuse of notation, we identify $\mathbf{s}-\mathbf{r} \geqslant 0$ with any of its representatives in $\operatorname{mod} E_k$, and $\mathrm{Gr}_{e_k-\mathbf{r}}(\mathbf{s}-\mathbf{r}) = \varnothing$ if $e_k - \mathbf{r}$ or $\mathbf{s}-\mathbf{r}$ are not nonnegative). Hence, using the easy fact that $\mathrm{Gr}_{\mathbf{e}}(X)$ is the disjoint union of the $\widetilde{Z}_{\mathbf{e},\mathbf{r},\mathbf{s}}(X)$, we obtain, as every considered irreducible representation is of dimension 1,

$$\chi\left(\mathrm{Gr}_{\mathbf{e}}(X)\right) = \sum_{\mathbf{r},\mathbf{s}\in C_k} \binom{\mathbf{s}-\mathbf{r}}{e_k-\mathbf{r}} \chi\left(Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right).$$

where, for any $\mathbf{r}_1, \mathbf{r}_2 \in C_k$,

$$\binom{\mathbf{r}_1}{\mathbf{r}_2} = \prod_{\rho\in \mathrm{ind}_k} \binom{\mathbf{r}_{1,\rho}}{\mathbf{r}_{2,\rho}}$$

Then, substituting this expression in the definition of F_X, we obtain:

$$\begin{aligned}
F_X(y) &= \sum_{\mathbf{e}\in C}\left[\sum_{\mathbf{r},\mathbf{s}\in C_k}\binom{\mathbf{s}-\mathbf{r}}{e_k-\mathbf{r}}\chi\left(Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right)\right]y^{\mathbf{e}} \\
&= \sum_{\substack{\mathbf{e}'\in\bigoplus_{i\neq k}C_i\\ \mathbf{r},\mathbf{s}\in C_k}}\chi\left(Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right)y^{\mathbf{e}'}\sum_{e_k\in C_k}\binom{\mathbf{s}-\mathbf{r}}{e_k-\mathbf{r}}y_k^{e_k} \\
&= \sum_{\substack{\mathbf{e}'\in\bigoplus_{i\neq k}C_i\\ \mathbf{r},\mathbf{s}\in C_k}}\chi\left(Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right)y^{\mathbf{e}'+\mathbf{r}}(1+y_k)^{\mathbf{s}-\mathbf{r}}.
\end{aligned}$$

Now, as in [DWZ1], we have easily that

$$Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X) = Z_{\mathbf{e}',\overline{\mathbf{r}},\overline{\mathbf{s}}}(X')$$

where

$$\overline{\mathbf{r}} = \left[\mathbf{e}'\otimes_{\overline{E}_k}A^*E_k\right]-h_k-\mathbf{s} \quad\text{and}\quad \overline{\mathbf{s}} = \left[\mathbf{e}'\otimes_{\overline{E}_k}AE_k\right]-h'_k-\mathbf{r}.$$

Using this, one gets

$$\begin{aligned}
(1+y'_k)^{h'_k}F_{X'}(y') &= \sum_{\substack{\mathbf{e}'\in\bigoplus_{i\neq k}C_i\\ \overline{\mathbf{r}},\overline{\mathbf{s}}\in C_k}}\chi\left(Z_{\mathbf{e}',\overline{\mathbf{r}},\overline{\mathbf{s}}}(X')\right)y'^{\mathbf{e}'+\overline{\mathbf{r}}}(1+y'_k)^{h'_k+\overline{\mathbf{s}}-\overline{\mathbf{r}}} \\
&= \sum_{\substack{\mathbf{e}'\in\bigoplus_{i\neq k}C_i\\ \mathbf{r},\mathbf{s}\in C_k}}\chi\left(Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right)y'^{\mathbf{e}'}y_k^{-\overline{\mathbf{s}}-h'_k}(1+y_k)^{h'_k+\overline{\mathbf{s}}-\overline{\mathbf{r}}} \\
&= \sum_{\substack{\mathbf{e}'\in\bigoplus_{i\neq k}C_i\\ \mathbf{r},\mathbf{s}\in C_k}}\chi\left(Z_{\mathbf{e}',\mathbf{r},\mathbf{s}}(X)\right)y^{\mathbf{e}'+\mathbf{r}}(1+y_k)^{h_k+\mathbf{s}-\mathbf{r}} \\
&= (1+y_k)^{h_k}F_X(y)
\end{aligned}$$

(iii) As $g_k = h_k - h'_k$, $g'_k = -g_k$. If $j \neq k$, the equality we want to prove becomes, using again $g_k = h_k - h'_k$,

$$[\ker\gamma'_j] - \left[\ker\beta'_k\otimes_{E_k}A_{kj}\right] = [\ker\gamma_j] - \left[\ker\beta_k\otimes_{E_k}A^*_{jk}\right]$$

and, up to a possible exchange of (A,S,X,V) and (A',S',X',V'), we can suppose that $A_{kj}=0$ (because A is 2-acyclic) and therefore, we have to prove that

$$[\ker\gamma'_j] = [\ker\gamma_j] - \left[\ker\beta_k\otimes_{E_k}A^*_{jk}\right].$$

Let

$$(\widetilde{A},\widetilde{S},\widetilde{X},\widetilde{V}) = \widetilde{\mu}_k(A,S,X,V)$$

in such a way that (A',S') is right-equivalent to $(\widetilde{A},\widetilde{S})_{\mathrm{red}}$. In this setting, one will prove that

$$[\ker\widetilde{\gamma}_j] = [\ker\gamma_j] - \left[\ker\beta_k\otimes_{E_k}A^*_{jk}\right].$$

We can decompose

$$X_{\mathrm{out}}(j) = X\otimes_E A^*E_j = X(k)\otimes_{E_k}A^*_{jk}\oplus X\overline{E}_k\otimes_{\overline{E}_k}\overline{E}_kA^*E_j$$

and we get

$$\widetilde{X}_{\rm out}(j) = X_{\rm out}(k) \otimes_{E_k} A^*_{jk} \oplus X\overline{E}_k \otimes_{\overline{E}_k} \overline{E}_k A^* E_j$$

and

$$\widetilde{X}_{\rm in}(j) = \widetilde{X}(k) \otimes_{E_k} \widetilde{A}_{kj} \oplus X_{\rm in}(j) = X'(k) \otimes_{E_k} A^*_{jk} \oplus X_{\rm in}(j).$$

Along these decompositions, one has:

$$\gamma_j = \begin{pmatrix} \psi \circ (\beta_k \otimes_{E_k} A^*_{jk}) & \eta \end{pmatrix} \quad \text{and} \quad \widetilde{\gamma}_j = \begin{pmatrix} \alpha'_k \otimes_{E_k} A^*_{jk} & 0 \\ \psi & \eta \end{pmatrix}$$

where $\psi : X_{\rm out}(k) \otimes_{E_k} A^*_{jk} \to X_{\rm in}(j)$ and $\eta : X\overline{E}_k \otimes_{\overline{E}_k} \overline{E}_k A^* E_j \to X_{\rm in}(j)$ are two E_j-modules morphisms (basically speaking, these two morphisms encode the part of γ_j which is not modified by the mutation at k). Using definitions of section 8, we get easily that $\ker \alpha'_k = \operatorname{im} \beta_k$ and we get an exact sequence of E_j-modules:

$$0 \to \ker \beta_k \otimes_{E_k} A^*_{jk} \oplus \{0\} \to \ker \gamma_i \xrightarrow{f} \ker \widetilde{\gamma}_i \to 0$$

where, along the previous decompositions

$$f(u, v) = ((\beta_k \otimes_{E_k} A^*_{jk})u, v).$$

This short exact sequence implies that

$$[\ker \widetilde{\gamma}_j] = [\ker \gamma_j] - \left[\ker \beta_k \otimes_{E_k} A^*_{jk}\right].$$

To finish, it remains to prove that $[\ker \widetilde{\gamma}_j] = [\ker \gamma'_j]$. The proof is the same than in [DWZ1]. $\square$

Definition 9.15. For any GSPDR (A, S, X, V), we define in the following way the reduced $\mathbf{g}$-vectors, $\mathbf{h}$-vectors and F-polynomials:

- for $i \in I$, let $\check{\mathbf{g}}_{X,V} = (\check{g}_i)_{i\in I}$ defined by $\check{g}_i = \dim_K g_i$ where $(g_i)_{i\in I} = \mathbf{g}_{X,V}$;
- for $i \in I$, let $\check{\mathbf{h}}_{X,V} = (\check{h}_i)_{i\in I}$ defined by $\check{h}_i = \dim_K h_i$ where $(h_i)_{i\in I} = \mathbf{h}_{X,V}$;
- $\check{F}_X = \Phi_{Y\to Z}(F_X)$ where $(Y_i)_{i\in \rm irr}$ and $(Z_i)_{i\in I}$ are families of indeterminates.

Corollary 9.16. *Let (A, S, X, V) be a GSPDR such that (A, S) is non-degenerate and locally free. Let $k \in I$. Denote*

$$(A', S', X', V') = \mu_k(A, S, X, V).$$

Suppose also that the Y-seed $(z', B(A'))$ is obtained from $(z, B(A))$ by the mutation at k. Denote $\check{\mathbf{g}}_{X,V} = (\check{g}_i)_{i\in I}$, $\check{\mathbf{g}}_{X',V'} = (\check{g}'_i)_{i\in I}$, $\check{\mathbf{h}}_{X,V} = (\check{h}_i)_{i\in I}$ and $\check{\mathbf{h}}_{X',V'} = (\check{h}'_i)_{i\in I}$. We also denote by $(b_{ij})_{i,j\in I}$ the coefficients of $B(A)$. Then

(i) $\forall i \in I, \check{g}_i = \check{h}_i - \check{h}'_i$;

(ii) one has

$$(z_k + 1)^{\check{h}_k} \check{F}_X(z) = (z'_k + 1)^{\check{h}'_k} \check{F}_{X'}(z');$$

(iii) for any $j \in I$,

$$\check{g}'_j = \begin{cases} -\check{g}_j & \text{if } j = k \\ \check{g}_j + \max(0, b_{jk})\check{g}_k - b_{jk}\check{h}_k & \text{if } j \neq k; \end{cases}$$

(iv) if $F_X \in \mathbb{Q}_+(Y)$*, then* $\check{F}_X \in \mathbb{Q}_+(Z)$*. Then* $\check{\mathbf{h}}_X$ *and* $\check{F}_X$ *are related by the following formula:*

$$Z_0^{\check{\mathbf{h}}_X} = \check{F}_X|_{\mathrm{Trop}(Z_0)} \left(Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})} \right)_{i \in I}.$$

Proof. The points (i) and (iii) are immediate consequences of proposition 9.14. To prove (ii), it is enough to apply $\Phi_{y \to z}$ to the analogous identity in proposition 9.14 (for any extended free Y-seed $(y,(A,S)))$ and then apply proposition 9.13. For (iv), remark that for any $(i,\rho) \in \mathrm{irr}$,

$$\Phi_{Y_0 \to Z_0} \left(Y_{0,i,\rho}^{-1} Y_0^{[\rho \otimes_{E_i} E_i A^*]} \right) = Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})}$$

is independent of ρ and therefore, it is easy to see that

$$\begin{aligned} &\check{F}_X|_{\mathrm{Trop}(Z_0)} \left(Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})} \right)_{i \in I} \\ =&\Phi_{Y_0 \to Z_0} \left(F_X|_{\mathrm{Trop}(Y_0)} \left(Y_{0,i,\rho}^{-1} Y_0^{[\rho \otimes_{E_i} E_i A^*]} \right)_{(i,\rho) \in I} \right) \\ =&\Phi_{Y_0 \to Z_0} \left(Y_0^{\mathbf{h}_X} \right) = Z_0^{\check{\mathbf{h}}_X} \end{aligned}$$

using proposition 9.6. □

In [FZ2], (see also [DWZ1, §2]), Fomin and Zelevinsky defined the notions of the F-polynomials and the $\mathbf{g}$-vectors associated to a sequence of mutation. More precisely, for a skew-symmetrizable matrix B (which will play the role of an initial seed), a sequence of indices $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ and $k \in I$, they define a polynomial $F_{k;\mathbf{i}}^B \in \mathbb{Z}[Z_i]_{i \in I}$ and a vector $\mathbf{g}_{k;\mathbf{i}}^B \in \mathbb{Z}^I$.

Definition 9.17. Let (A,S) be a non-degenerate GSP and $\mathbf{i} = (i_1, \ldots, i_n)$ be in I^n and V an E-module. We denote

$$\left(A_{V;\mathbf{i}}^{A,S}, S_{V;\mathbf{i}}^{A,S}, X_{V;\mathbf{i}}^{A,S}, V_{V;\mathbf{i}}^{A,S} \right) = \mu_{i_1} \mu_{i_2} \ldots \mu_{i_n} \left(\mu_{i_n} \ldots \mu_{i_2} \mu_{i_1} (A,S), 0, V \right).$$

Remark that $\left(A_{V;\mathbf{i}}^{A,S}, S_{V;\mathbf{i}}^{A,S} \right)$ is right-equivalent to (A,S).

Thus, we can adapt theorem [DWZ1, theorem 5.1]:

Theorem 9.18. *Let* (A,S) *be a non-degenerate locally free GSP. Let* $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$*,* $k \in I$ *and* $\rho \in \mathrm{irr}_k$*. Then*

$$\mathbf{g}_{k;\mathbf{i}}^{B(A)} = \check{\mathbf{g}}_{X_{\rho;\mathbf{i}}^{A,S}, V_{\rho;\mathbf{i}}^{A,S}} \quad \text{and} \quad F_{k;\mathbf{i}}^{B(A)} = \check{F}_{X_{\rho;\mathbf{i}}^{A,S}}.$$

Proof. With corollary 9.16, it is the same proof as in [DWZ1]. □

We get also this following, analogous to [DWZ1, corollary 5.3]:

Corollary 9.19. *In the situation of theorem 9.18, suppose that* $F_{k;\mathbf{i}}^{B(A)} \neq 1$*, hence* $X_{\rho;\mathbf{i}}^{A,S} \neq \{0\}$ *and* $V_{\rho;\mathbf{i}}^{A,S} = \{0\}$ *(see proposition 8.8). Let* $x_{k;\mathbf{i}}^{B(A)}$ *be the corresponding cluster variable in the coefficient-free cluster algebra. In other terms*

$$\left(\left(x_{i;\mathbf{i}}^{B(A)} \right)_{i \in I}, B' \right) = \mu_{i_n} \ldots \mu_{i_2} \mu_{i_1} \left((x_i)_{i \in I}, B(A) \right).$$

Then we have the following cluster character formula:

$$x_{k;\mathbf{i}}^{B(A)} = \prod_{i \in I} x_i^{-d_i} \sum_{\mathbf{e} \in C} \chi\left(\mathrm{Gr}_{\mathbf{e}}(X)\right) \prod_{i \in I} x_i^{-\operatorname{rk}\gamma_i + \sum_{j \in I}(\max(0,b_{ij})e_j + \max(0,-b_{ij})(d_j - e_j))}$$

where $X = X_{\rho;\mathbf{i}}^{A,S}$, $d_i = \dim_K X(i)$ *and* $e_i = \dim_K \mathbf{e}_i$.

10. $\mathcal{E}$-invariant

The aim of this part is analogous to [DWZ1, §7, §8]. Let (A, S, X, V) and (A, S, X', V') be two GSPDRs with the same non-degenerate GSP. We denote:

$$\langle X, X' \rangle = \dim_K \operatorname{Hom}_{\mathcal{P}(A,S)}(X, X').$$

Define the three following integer functions:

$$\mathcal{E}^{\mathrm{inj}}(X, V; X', V') = \langle X, X' \rangle + \left([X] | \mathbf{g}_{X',V'}\right)$$

$$\mathcal{E}^{\mathrm{sym}}(X, V; X', V') = \mathcal{E}^{\mathrm{inj}}(X, V; X', V') + \mathcal{E}^{\mathrm{inj}}(X', V'; X, V)$$

$$\mathcal{E}(X, V) = \mathcal{E}^{\mathrm{inj}}(X, V; X, V) = \frac{\mathcal{E}^{\mathrm{sym}}(X, V; X, V)}{2}$$

where $[X] \in C$ is the class of X seen as an E-module, and, for $\mathbf{e}, \mathbf{e}' \in C$ (resp. $\mathbf{e}, \mathbf{e}' \in C_k$ for $k \in I$),

$$\left(\mathbf{e} | \mathbf{e}'\right) = \sum_{\substack{i \in \mathrm{irr} \\ (\text{resp. } i \in \mathrm{irr}_k)}} \mathbf{e}_i \mathbf{e}'_i.$$

Then, we get, with the same proof as [DWZ1, theorem 7.1]:

Theorem 10.1. *We have, for any* $k \in I$,

$$\begin{aligned} &\mathcal{E}^{\mathrm{inj}}\left(\mu_k(X, V); \mu_k(X', V')\right) - \mathcal{E}^{\mathrm{inj}}\left(X, V; X', V'\right) \\ &= \left(\mathbf{h}_{\mu_k(X,V),k} | \mathbf{h}_{X',V',k}\right) - \left(\mathbf{h}_{X,V,k} | \mathbf{h}_{\mu_k(X',V'),k}\right). \end{aligned}$$

In particular, $\mathcal{E}^{\mathrm{sym}}$ *and* $\mathcal{E}$ *are stable under mutations.*

Proof. The only difference with [DWZ1] is that computations have to be done in the Grothendieck groups. Moreover, we have to worry about the skew-symmetrizability: with our convention, informally speaking, all b_{ik} should be replaced by $-b_{ki}$ in the proof of [DWZ1]). For example,

$$\sum_{i \in I} \max(0, b_{ik}) \dim_K X(i)$$

in [DWZ1] will be replaced here by $[X \otimes_E A^* E_k]$ whose dimension is

$$\sum_{i \in I} \max(0, -b_{ki}) \dim_K X(i)$$

if the GSP is locally free and $B = B(A)$. □

We get also the following analogous of [DWZ1, corollary 7.2]:

Corollary 10.2. *If* (X, V) *is obtained by a sequence of mutations from a negative decorated representation* $(\{0\}, V)$ *then* $\mathcal{E}(X, V) = 0$.

We denote by A^{op} the (E,E)-bimodule whose underlying vector space is A and whose bimodule structure is given by $g \cdot a^{\mathrm{op}} \cdot h = (h^{-1} \cdot a \cdot g^{-1})^{\mathrm{op}}$ if $g \in \Gamma_i$ and $h \in \Gamma_j$ for some $i, j \in I$ and $\mathrm{op} : A \to A^{\mathrm{op}}$ comes from the identity of A. It is then easy to extend op to an anti-isomorphism of algebras $E\langle\langle A\rangle\rangle \to E\langle\langle A^{\mathrm{op}}\rangle\rangle$. Thus, (X^*, V^*) is a decorated representation of the GSP $(A^{\mathrm{op}}, S^{\mathrm{op}})$ on the ring E, where for each $i \in I$, X_i^* is contragredient to X_i, V_i^* is contragredient to V_i and a^{op} acts on X^* as the transpose of a for every $a \in A$. Thus, one gets the analogous of [DWZ1, proposition 7.3]:

Proposition 10.3. *We have $\mathcal{E}(X^*, V^*) = \mathcal{E}(X, V)$.*

Proof. As for any $i \in I$, the characteristic of K does not divide $\#\Gamma_i$, we have an isomorphism of right E-modules

$$\begin{aligned} (X \otimes_E A)^* &\to X^* \otimes_E A^{*\,\mathrm{op}} \simeq X^* \otimes_E A^{\mathrm{op}\,*} \\ f &\mapsto \sum_{\substack{x \in \mathcal{B}_X \\ a \in \mathcal{B}_A}} f(x \otimes a) x^* \otimes a^{*\,\mathrm{op}} \\ \left(x \otimes a \mapsto \sum_{i \in I} \sum_{g \in \Gamma_i} \frac{\varphi(xg)\psi(g^{-1}a)}{\#\Gamma_i} \right) &\leftarrow \varphi \otimes \psi^{\mathrm{op}} \end{aligned}$$

which does not depend of the bases $\mathcal{B}_X$ and $\mathcal{B}_A$ of X and A. Thus, we have, as in [DWZ1],

$$\begin{aligned} \mathcal{E}(X, V) =& \langle X, X \rangle + ([X] | [X \otimes_E A^*]) + \left([X] \,\middle|\, [V] - [X] - \left[\bigoplus_{i \in I} \operatorname{im} \gamma_i \right] \right) \\ =& \langle X, X \rangle + ([X \otimes_E A] | [X]) + \left([X] \,\middle|\, [V] - [X] - \left[\bigoplus_{i \in I} \operatorname{im} \gamma_i \right] \right) \\ =& \langle X^*, X^* \rangle + ([(X \otimes_E A)^*] | [X^*]) \\ & + \left([X^*] \,\middle|\, [V^*] - [X^*] - \left[\bigoplus_{i \in I} \operatorname{im} \gamma_i^* \right] \right) \\ =& \langle X^*, X^* \rangle + ([X^* \otimes_E A^{\mathrm{op}\,*}] | [X^*]) \\ & + \left([X^*] \,\middle|\, [V^*] - [X^*] - \left[\bigoplus_{i \in I} \operatorname{im} \gamma_i^* \right] \right) \\ =& \mathcal{E}(X^*, V^*) \end{aligned}$$

where we used that

$$\begin{aligned} ([X] | [X \otimes_E A^*]) &= \dim_K \operatorname{Hom}_E(X, X \otimes_E A^*) \\ &= \dim_K \operatorname{Hom}_E(X \otimes_E A, X) = ([X \otimes_E A] | [X]) \,. \quad \square \end{aligned}$$

Hence, the following theorem has the same proof as [DWZ1, theorem 8.1] (note that all [DWZ1, §10] can be easily adapted in this case):

Theorem 10.4. *The $\mathcal{E}$-invariant satisfies*

$$\mathcal{E}(X, V) \geqslant \left(\left[\bigoplus_{i \in I} \ker \beta_i \right] \,\middle|\, \left[\bigoplus_{i \in I} \frac{\ker \gamma_i}{\operatorname{im} \beta_i} \right] \right) + ([X] | [V]) \,.$$

Then, we obtain the analogous of [DWZ1, corollary 8.3]:

Corollary 10.5. *If* $\mathcal{E}(X, V) = 0$ *then for each* $(k, \rho) \in \mathrm{irr}$*,*

(i) either $[M_k]_\rho = 0$ *or* $[V_k]_\rho = 0$*;*
(ii) either $[\ker \gamma_k]_\rho = 0$ *or* $[\ker \gamma_k]_\rho = [\mathrm{im}\, \beta_k]_\rho$*.*

11. Applications to cluster algebras

We conclude here that the following conjectures of [FZ2] are true for skew-symmetrizable integer matrix which can be obtained from a non-degenerate GSP with abelian groups. In particular, every matrix of the form DS where D is diagonal with integer coefficients and S is skew-symmetric with integer coefficients can be obtained in view of section 7. Every exchange matrix corresponding to the situation described in [Dem] (in particular every acyclic ones) can also be raised. Let B be such a skew-symmetrizable integer matrix. We suppose moreover that some (A, S) is fixed satisfying the hypothesis of section 9 such that $B(A) = B$.

Proposition 11.1 ([FZ2, conjecture 5.4])**.** *For any* $\mathbf{i} \in I^n$ *and* $k \in I$*,* $F^B_{k;\mathbf{i}}$ *has constant term* 1*.*

Proposition 11.2 ([FZ2, conjecture 5.5])**.** *For any* $\mathbf{i} \in I^n$ *and* $k \in I$*,* $F^B_{k;\mathbf{i}}$ *has a maximum monomial for divisibility order with coefficient* 1*.*

These first two are immediate, as in [DWZ1, §9].

Proposition 11.3 ([FZ2, conjecture 7.12])**.** *For any* $\mathbf{i} \in I^n$*,* $k \in I$*, we denote by* $k\mathbf{i}$ *the concatenation of* (k) *and* $\mathbf{i}$*. Let* $j \in I$ *and* $(g_i)_{i\in I} = \mathbf{g}^B_{j;\mathbf{i}}$ *and* $(g'_i)_{i\in I} = \mathbf{g}^{\mu_k(B)}_{j;k\mathbf{i}}$*. Then we have, for any* $i \in I$*,*

$$g'_i = \begin{cases} -g_i & \text{if } i = k; \\ g_i + \max(0, b_{ik}) g_k - b_{jk} \min(g_k, 0) & \text{if } i \neq k. \end{cases}$$

Proof. We need here to add some trick to the proof of [DWZ1, §9]. Indeed, we need to prove, as in [DWZ1], that

$$\min(0, g_k) = h_k.$$

But what we obtain by using corollary 10.5 is

$$\min(0, g_{k,\rho}) = h_{k,\rho}$$

for any $\rho \in \mathrm{irr}_k$. Moreover, we have, as seen before,

$$g_k = \sum_{\rho \in \mathrm{irr}_k} g_{k,\rho} \quad \text{and} \quad h_k = \sum_{\rho \in \mathrm{irr}_k} h_{k,\rho}$$

and therefore, what we need is equivalent to the fact that the $g_{k,\rho}$ are of the same sign. We will prove this with an indirect method. Retaining the notation of definition 9.17, we get

$$X^{A,S}_{E_j;\mathbf{i}} = \sum_{\rho \in \mathrm{irr}_j} X^{A,S}_{\rho;\mathbf{i}}$$

and therefore, by linearity of $\mathbf{g}$,

$$\mathbf{g}_{X^{A,S}_{E_j;\mathbf{i}}} = \sum_{\rho \in \mathrm{irr}_j} \mathbf{g}_{X^{A,S}_{\rho;\mathbf{i}}}.$$

Hence, we get:

$$(\#\Gamma_j)g_k = \dim_K \left[\mathbf{g}_{X^{A,S}_{E_j;\mathbf{i}}}\right]_k.$$

In the same way,

$$(\#\Gamma_j)h_k = \dim_K \left[\mathbf{h}_{X^{A,S}_{E_j;\mathbf{i}}}\right]_k.$$

Moreover, by an immediate induction using proposition 9.14, as $[E_j]$ is the class of a free E_j-module, $\left[\mathbf{g}_{X^{A,S}_{E_j;\mathbf{i}}}\right]_k$ and $\left[\mathbf{h}_{X^{A,S}_{E_j;\mathbf{i}}}\right]_k$ are also free and therefore, their coefficients in term of the irreducible representations of E_k are of the same sign. Hence, we obtain, by adding these components

$$\min(0, (\#\Gamma_j)g_k) = (\#\Gamma_j)h_k$$

and the rest follows as in [DWZ1]. Note that it implies also that the $g_{k,\rho}$ are of the same sign. □

The three following propositions have the same proof than in [DWZ1, §9]:

Proposition 11.4 ([FZ2, conjecture 6.13])**.** *For any $\mathbf{i} \in I^n$, the vectors $\mathbf{g}^B_{i;\mathbf{i}}$ for $i \in I$ are sign-coherent. In other terms, for $i, i', j \in I$, the j-th components of $\mathbf{g}^B_{i;\mathbf{i}}$ and $\mathbf{g}^B_{i';\mathbf{i}}$ have the same sign.*

Proposition 11.5 ([FZ2, conjecture 7.10(2)])**.** *For any $\mathbf{i} \in I^n$, the vectors $\mathbf{g}^B_{i;\mathbf{i}}$ for $i \in I$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^I$.*

Proposition 11.6 ([FZ2, conjecture 7.10(1)])**.** *For any $\mathbf{i}, \mathbf{i}' \in I^n$, if we have*

$$\sum_{i\in I} a_i \mathbf{g}^B_{i;\mathbf{i}} = \sum_{i\in I} a'_i \mathbf{g}^B_{i;\mathbf{i}'}$$

for some nonnegative integers $(a_i)_{i\in I}$ and $(a'_i)_{i\in I}$, then there is a permutation $\sigma \in \mathfrak{S}_I$ such that for every $i \in I$,

$$a_i = a'_{\sigma(i)} \quad \text{and} \quad a_i \neq 0 \Rightarrow \mathbf{g}^B_{i;\mathbf{i}} = \mathbf{g}^B_{\sigma(i);\mathbf{i}'} \quad \text{and} \quad a_i \neq 0 \Rightarrow F^B_{i;\mathbf{i}} = F^B_{\sigma(i);\mathbf{i}'}.$$

In particular, $F^B_{i;\mathbf{i}}$ is determined by $\mathbf{g}^B_{i;\mathbf{i}}$.

12. An example and a counterexample

The aim of this part is to show an example where the technique shown in the previous sections works and a counterexample where there is no non-degenerate potential.

Suppose here that $K = \mathbb{C}$. We fix $\Gamma_1 = \Gamma_2$ to be the trivial group and $\Gamma_3 = \mathbb{Z}/2\mathbb{Z}$. We take $A_{12} = \mathbb{C}$ and $A_{23} = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$, the other A_{ij} vanishing. Then A is acyclic and therefore $S = 0$ is a non-degenerate potential, in view of section 7. Moreover,

$$B(A) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

which is of type C_3. Its exchange graph is given on figure 1 where the small dots $(\cdot)$ symbolize vertices with trivial group and big dots $(\bullet)$ symbolize

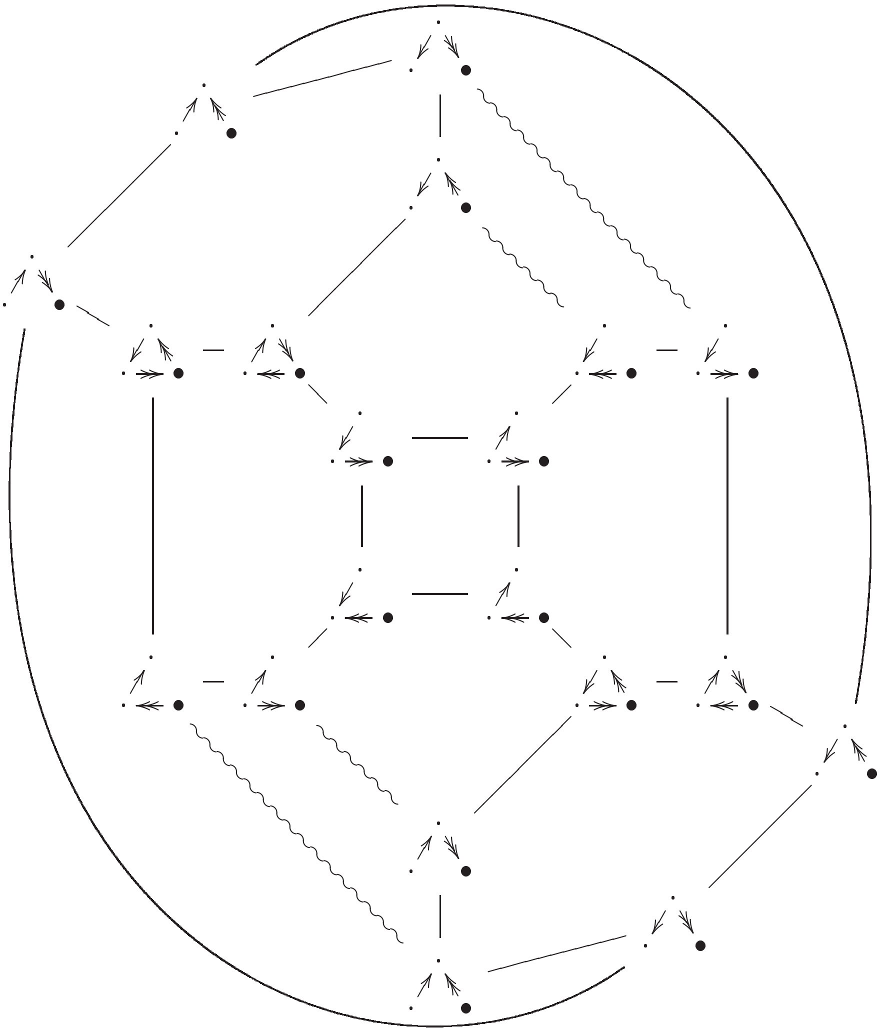
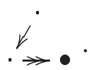


Figure 1. Exchange graph of type B_3

vertices with group $\mathbb{Z}/2\mathbb{Z}$. Simple arrows symbolize $\mathbb{C}$ and double arrows symbolize $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$. Thus, (A, S) will be symbolized by



Finally, wave lines ($\sim$) symbolize mutations composed with the exchange of vertices 1 and 2.

Now, we will compute explicitly $F^B_{3;213}$ and $\mathbf{g}^B_{3;213}$. We will follow the construction of section 9. According to the exchange graph,

$\mu_3\mu_1\mu_2(A,0) = \Big($  $, 0\Big) = (A', S').$

Let ρ be one the two irreducible modules over $\mathbb{Z}/2\mathbb{Z}$. Then

$$\mu_3(A', S', 0, \rho) = \left(\begin{tikzcd}[row sep=small, column sep=small] & \cdot \arrow[dl] & \\ \cdot & & \bullet \arrow[ul, two heads] \end{tikzcd}, 0, \begin{tikzcd}[row sep=small, column sep=small] & 0 & \\ 0 & & \rho \end{tikzcd}, 0 \right)$$

$$\mu_1\mu_3(A', S', 0, \rho) = \left(\begin{tikzcd}[row sep=small, column sep=small] & \cdot \arrow[dr, two heads] & \\ \cdot \arrow[ur] & & \bullet \arrow[ll, two heads] \end{tikzcd}, \dots, \begin{tikzcd}[row sep=small, column sep=small] & \mathbb{C} \arrow[dr, two heads] & \\ 0 & & \rho \end{tikzcd}, 0 \right)$$

$$\mu_2\mu_1\mu_3(A', S', 0, \rho) = \left(\begin{tikzcd}[row sep=small, column sep=small] & \cdot \arrow[dl] & \\ \cdot \arrow[rr, two heads] & & \bullet \end{tikzcd}, \dots, \begin{tikzcd}[row sep=small, column sep=small] & \mathbb{C} \arrow[dl] & \\ \mathbb{C} \arrow[rr, two heads] & & \rho \end{tikzcd}, 0 \right)$$

(the arrows are obvious) and therefore,

$$X^B_{\rho;213} = \begin{tikzcd}[row sep=small, column sep=small] & \mathbb{C} \arrow[dl] & \\ \mathbb{C} \arrow[rr, two heads] & & \rho \end{tikzcd}$$

which induces that:

$$F_{X^B_{\rho;213}} = 1 + Y_\rho + Y_2 Y_\rho + Y_1 Y_2 Y_\rho$$

and therefore

$$\check{F}_{X^B_{\rho;213}} = 1 + Y_3 + Y_2 Y_3 + Y_1 Y_2 Y_3.$$

Moreover,

$$\mathbf{g}_{X^B_{\rho;213}} = \begin{pmatrix} 0 \\ 0 \\ -\rho \end{pmatrix}$$

and therefore

$$\check{\mathbf{g}}_{X^B_{\rho;213}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

It is easy to check by hand that these coincide with $F^B_{3;213}$ and $\mathbf{g}^B_{3;213}$ obtained for example by formulas of [DWZ1, §2].

Let now B be the matrix defined by

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will show that there is no non-degenerate locally free GSP with mutation matrix B. Suppose that $(I, (\Gamma_i), A, S)$ is a non-degenerate GSP with mutation matrix B. Then, $\Gamma_1, \dots, \Gamma_5$ have the same cardinal which is two times the one of Γ_6. Applying μ_3 followed by μ_5 create 2-cycles and implies, in view of proposition 4.10, that

$$A_{23} \otimes_{E_3} A_{31} \simeq (A_{15} \otimes_{E_5} A_{52})^*.$$

In the same way, applying μ_4 followed by μ_5 implies that

$$A_{24} \otimes_{E_4} A_{41} \simeq \left(A_{15} \otimes_{E_5} A_{52}\right)^* .$$

With the same type of argument, applying μ_3, μ_4 and μ_6 implies that

$$\left(A_{23} \otimes_{E_3} A_{31}\right)^{\oplus 2} \simeq A_{24} \otimes_{E_4} A_{41} \oplus A_{23} \otimes_{E_3} A_{31} \simeq \left(A_{16} \otimes_{E_6} A_{62}\right)^* .$$

As all considered groups are semisimple, it is easy to see that the (E_1, E_6)-bimodule A_{16} can be decomposed as a direct sum of the form

$$A_{16} = \bigoplus_{i=1}^{m} r_i \otimes_K s_i$$

where the r_i are irreducible left E_1-modules and the s_i are irreducible right E_6-modules. Moreover, the $r_i \otimes_K s_i$ are irreducible bimodule and satisfy, because of B,

$$\forall r \in \mathrm{irr}_1, \sum_{i \mid r_i \simeq r} \dim_K s_i = \dim_K r \text{ and } \forall s \in \mathrm{irr}_6, \sum_{i \mid s_i \simeq s} \dim_K r_i = 2 \dim_K s.$$

Thus, there are exactly two indices which can be supposed to be 1 and 2 such that s_1, s_2 are trivial and r_1 and r_2 are of dimension 1 and appear only one time in the sequence (r_i). In the same way,

$$A_{62} = \bigoplus_{i=1}^{n} t_i \otimes_K u_i$$

with

$$\forall t \in \mathrm{irr}_6, \sum_{i \mid t_i \simeq t} \dim_K u_i = 2 \dim_K t \text{ and } \forall u \in \mathrm{irr}_2, \sum_{i \mid u_i \simeq u} \dim_K t_i = \dim_K u.$$

Thus, there are exactly two indices which can be supposed to be 1 and 2 such that t_1, t_2 are trivial and the u_1 and u_2 are of dimension 1 and appear only one time in the sequence (u_j). Hence,

$$\left(A_{16} \otimes A_{62}\right)^* = \bigoplus_{\substack{i=1 \\ s_i \simeq t_j^*}}^{m} \bigoplus_{j=1}^{n} \left(u_j^* \otimes_K r_i^*\right)^{\dim_K s_i}$$

contains $u_1^* \otimes r_1^* \oplus u_1^* \otimes r_2^* \oplus u_2^* \otimes r_1^* \oplus u_2^* \otimes r_2^*$ as the only summands containing u_1^*, u_2^*, r_1^* and r_2^*. Finally, $\left(A_{16} \otimes A_{62}\right)^*$ can not be decomposed as a direct sum of two times the same bimodule, which is a contradiction.

Acknowledgments

The author would like to thank Bernhard Keller, Sefi Ladkani, Bernard Leclerc and Dong Yang for discussions in relation to this subject.

References

[Dem] Laurent Demonet. Categorification of skew-symmetrizable cluster algebras. arXiv: 0909.1633.

[DR] Vlastimil Dlab and Claus Michael Ringel. Indecomposable representations of graphs and algebras. *Mem. Amer. Math. Soc.*, 6(173):v+57, 1976.

[DWZ1] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras. arXiv: 0904.0676.

[DWZ2] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)*, 14(1):59–119, 2008.

[FZ1] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[FZ2] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, 143(1):112–164, 2007.

Max Planck Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn
E-mail address: Laurent.Demonet@normalesup.org